ALGEBRAIC CURVES
SOLUTIONS SHEET 10

Unless otherwise specified, k is an algebraically closed field.

Exercise 1. For $n, d \geq 1$, let $V(d,n)$ the k-vector space of forms of degree d in $k[X_1, \dots, X_n]$.

(1) Compute $\dim_k(V(d,n))$ for $d \geq 1$ and $n = 1, 2, 3$. Can you find a formula for arbitrary n?

Set $n = 2$. Let $L_i,\ i \geq 1$ and $M_j,\ j \geq 1$ be two sequences of non-zero linear forms in $k[X,Y]$ such that $L_i \neq \lambda M_j$ for all $i, j \geq 1$, $\lambda \in k$. Consider $A_{ij} = L_1 \dots L_i M_1 \dots M_j,\ i, j \geq 0$ (if $i = 0$ or $j = 0$, the empty product is taken as 1).

(2) Show that $A_{ij},\ i + j = d,\ i, j \geq 0$ form a basis of $V(d, 2)$. (Hint: think of dehomogenizing the A_{ij} by setting $Y = 1$.)

Solution 1.

(1) As we have already seen throughout those exercise sheets, we can compute $\dim_k(V(d,n))$ in a combinatorial way, as $\dim_k(V(d,n)) = \{d_1 + \cdots + d_n = d \mid d_k \in [\![1, n]\!]\} = \binom{n+d-1}{n-1}$.

(2) As the L_i and M_j are homogeneous forms of degree 1 inside $k[X,Y]$, their vanishing loci define points in $\mathbb{P}^1$. Consider the sets

$$V = V_p(L_1 \cdots L_d) = V_p(L_1) \cup \cdots \cup V_p(L_d) \subseteq \mathbb{P}^1,$$
$$W = V_p(M_1 \cdots M_d) = V_p(M_1) \cup \cdots \cup V_p(M_d) \subseteq \mathbb{P}^1.$$

Then as $L_i \neq \lambda M_j$ for all i, j, λ, we have $V \cap W = \emptyset$.

Now suppose that

$$\sum_{i+j=d} \lambda_i A_{ij} = 0$$

for some $\lambda_0, \dots, \lambda_d \in k$. Choose now $v \in \mathbb{A}^2 \setminus \{0\}$ such that $V_p(M_1) = \{[v]\}$. Then we have

$$0 = \sum_{i+j=d} \lambda_i A_{ij}(v) = \lambda_d \underbrace{A_{d0}(v)}_{\neq 0}$$

and thus $\lambda_d = 0$ (as $[v] \notin V$). Now define $\widetilde{M_j} := M_{j+1}$, and define $\widetilde{A}_{ij}$ for $i + j = d - 1$ as $L_1 \cdots L_i \widetilde{M_1} \cdots \widetilde{M_j}$. Then we have

$$0 = \sum_{i+j=d,\ i>0} \lambda_i A_{ij} = M_1 \sum_{i+j=d-1} \lambda_{i+1} \widetilde{A}_{ij}.$$

As $M_1 \neq 0$, we obtain $0 = \sum_{i+j=d-1} \lambda_{i+1} \widetilde{A}_{ij}$, and thus by induction on d we obtain that $\lambda_i = 0$ for all i.

Exercise 2. Recall properties 1 to 9 of intersection numbers from the course (Thm. 4.5). Prove property 8 using only properties 1 to 7. (Hint: introduce a uniformizer ϖ of $\mathcal{O}_P(F)$ and rewrite the factorization $G = u\varpi^n$, $u \in \mathcal{O}_P(F)^\times$ in terms of polynomials in $k[X,Y]$.)

Solution 2. We need to show that for P a simple point of F, $I(P, F \cap G) = ord_P^F(G)$. We reduce to $P = (0,0)$. We have 3 distinct cases :

- $G(P) \neq 0$. Then $I(P, F \cap G) = ord_P^F(G) = 0$. Indeed, $G \in \mathcal{O}_P(F)^*$.
- P lies in a common component of F and G. By 1), $I(P, F \cap G) = \infty$. Moreover, $G = 0$ in $\Gamma(F)$ since F is irreducible. So $G = 0$ in $\mathcal{O}_P(F)$, so $ord_P^F(G) = \infty$.
- Otherwise, let ϖ be a uniformizer of $\mathcal{O}_P(F)$ and write $G = u\varpi^n$ with u a unit in $\mathcal{O}_P(F)$. If we write $u = (f + I(F))/(g + I(F))$ for some $f, g \in k[X,Y]$ with $f(P), g(P) \neq 0$ and $\varpi = (A + I(F))/(1 + I(F))$ with $a(P) = 0$ (we may always assume that the denominator is 1 by multiplying with a unit), then in $k[X,Y]$ we obtain
$$Gg = fA^n + hF$$
for some $h \in k[X,Y]$ (here we used that F is irreducible, so the localization map is injective). Using 2) and 6) we get $I(P, F \cap G) = I(P, F \cap Gg)$. Using 7) and again 2) and 6), we get : $I(P, F \cap gG) = I(P, F \cap fA^n) = I(P, F \cap A^n) = nI(P, F \cap A)$. Finally, notice that as in the first example of section 4.3, we have
$$\mathcal{O}_P(\mathbb{A}^2) \Big/ (F, A) \cong \mathcal{O}_P(F) \Big/ (A) = \mathcal{O}_P(F) \Big/ \mathfrak{m}_P \cong k,$$
where we used that $(A) = \mathfrak{m}_P$ inside $\mathcal{O}_P(F)$, as $\varpi = A/1$ is a uniormizer. Hence it follows that $I(P, F \cap A) = 1$ and thus $I(P, F \cap G) = n$.

Exercise 3. Compute the intersection numbers at $P = (0,0)$ of various pairs of the following curves:

- $A = Y - X^2$
- $B = Y^2 - X^3 + X$
- $C = Y^2 - X^3$
- $D = Y^2 - X^3 - X^2$
- $E = (X^2 + Y^2)^2 + 3X^2Y - Y^3$
- $F = (X^2 + Y^2)^3 - 4X^2Y^2$

Solution 3. To compute $I(P, F \cap G)$ for some curves F, G, the following strategy might be useful:

(1) Compute the tangents of F, G at P. If they don't share any tangents, then $I(P, F \cap G) = m_P(F)m_P(G)$ by 5).

(2) If F and G share some tangents, try to make one of the curves reducible using 7). For example, if F is of the form $Y^2 - P(X)$, you can replace every Y^2 in G by $P(X)$.
(3) You can also first multiply G by some H with $P \notin H$, which doesn't change the multiplicity by 6) and 2). For well chosen H, this might help you in the previous point.
(4) If one of the curves becomes irreducible, use 6) to split up the computation.
(5) Repeat.

Here is a table summarizing the solutions (if the result depends on the characteristic, the characteristic is indicated in subscript. For example, $I(P, A \cap E)$ is 6 if $\operatorname{char} k = 2$ and 4 otherwise, so the corresponding entry in the table is $6_2,\ 4_{\neq 2}$):

$I(P, - \cap -)$	A	B	C	D	E	F
A	∞	1	3	2	$6_2,\ 4_{\neq 2}$	$12_3,\ 6_{\neq 3}$
B		∞	2	2	3	$10_3,\ 6_{\neq 3}$
C			∞	4	$8_3,\ 7_{\neq 3}$	$12_2,\ 10_{\neq 2}$
D				∞	$8_2,\ 6_{\neq 2}$	$18_2,\ 8_{\neq 2}$
E					∞	$\infty_2,\ 18_5,\ 14_{\neq 2,5}$
F						∞

$A \cap B$: Note that P is a simple point on both A and B, with tangents Y resp. X. Hence by 5) we have

$$I(P, A \cap B) = 1.$$

$A \cap C$: We have

$$\begin{aligned} I(P, A \cap C) &\overset{7)}{=} I(P, (Y - X^2) \cap (X^4 - X^3)) \\ &\overset{6)}{=} I(P, (Y - X^2) \cap X^3) + I(P, (Y - X^2) \cap (X - 1)) \\ &\overset{6)+2)}{=} 3I(P, (Y - X^2) \cap X) \\ &\overset{7)}{=} 3. \end{aligned}$$

$A \cap D$: Note that the tangents to D at P are $X - Y$ and $X + Y$, and the tangent to A is X. As they are distinct, we obtain by 5) that

$$I(P, A \cap D) = m_P(A) m_P(D) = 2.$$

$A \cap E$: We have

$$\begin{aligned} I(P, A \cap E) &\overset{7)}{=} I(P, (Y - X^2) \cap ((X^2 + X^4)^2 + 3X^4 - X^6)) \\ &= I(P, (Y - X^2) \cap (X^8 + X^6 + 4X^4)). \end{aligned}$$

Using 5), one can see that the intersection number is 4 if $\operatorname{char} k \neq 2$ and 6 otherwise.

$A \cap F$: We have

$$I(P, A \cap F) \overset{7)}{=} I(P, (Y - X^2) \cap ((X^2 + X^4)^3 - 4X^6))$$
$$= I(P, (Y - X^2) \cap (X^{12} + 3X^{10} + 3X^8 - 3X^6)).$$

Using 5), one can see that the intersection number is 6 if $\operatorname{char} k \neq 3$ and 12 otherwise.

$B \cap C$: The tangent to B is X whereas the tangents to C are twice Y, so by 5) we obtain

$$I(P, B \cap C) = m_P(B)m_P(C) = 2.$$

$B \cap D$: Again the tangents of B and D are distinct at P, so we have

$$I(P, B \cap D) = m_P(B)m_P(D) = 2.$$

$B \cap E$: The tangents to E at P are Y and $Y \pm \sqrt{3}X$ and so they are different from the tangent to B. Hence by 5) we have

$$I(P, B \cap E) = m_P(B)m_P(E) = 3.$$

$B \cap F$: We have

$$I(P, B \cap F) \overset{7)}{=} I(P, (Y^2 - X^3 + X) \cap ((X^2 + X^3 - X)^3 - 4X^2(X^3 - X)))$$
$$= I(P, (Y^2 - X^3 + X) \cap ((\ldots)X^6 - 4X^5 + 3X^4 + 3X^3)).$$

Now if $\operatorname{char} k \neq 3$ we can factor out X^3 in the second curve and by 6) and 2) we obtain

$$\begin{aligned} I(P, B \cap F) &= I(P, (Y^2 - X^3 + X) \cap X^3) \\ &\overset{6)+7)}{=} 3I(P, Y^2 \cap X) \\ &= 6. \end{aligned}$$

On the other hand, if $\operatorname{char} k = 3$, we can factor out X^5 and with a similar computation we obtain $I(P, B \cap F) = 10$.

$C \cap D$: As the tangents to C are twice Y and to D it is $X \pm Y$, we obtain by 5) that

$$I(P, C \cap D) = m_P(C)m_P(D) = 4.$$

$C \cap E$: We have

$$(\bullet) \qquad I(P, C \cap E) \overset{7)}{=} I(P, (Y^2 - X^3) \cap ((X^2 + X^3)^2 + Y(3X^2 - X^3)))$$

Now if $\operatorname{char} k = 3$, the second curve is divisible by X^3, which gives using 6) and 7) that

$$\begin{aligned} I(P, C \cap E) &= I(P, (Y^2 - X^3) \cap X^3) + I(P, (Y^2 - X^3) \cap (X^3 + 2X^2 + X - Y)) \\ &\overset{6)+5)}{=} 6 + 2 \cdot 1 \\ &= 8, \end{aligned}$$

where we used 5) for the second intersection number, observing that the tangents are distinct.

On the other hand, if $\operatorname{char} k \neq 3$, the second curve in $(\bullet)$ is divisible by X^2, and using 6) and 7) we obtain

$$I(P, C \cap E) = 4 + I(P, (Y^2 - X^3) \cap (X^2(1+X)^2 + Y(3-X))).$$

To make the Y disappear in the first curve, we multiply it by $(3-X)^2$ (which doesn't change the multiplicity by 2) and 6)) and then we use 7):

$$\begin{aligned} I(P, C \cap E) &= 4 + I(P, (Y^2(3-X)^2 - X^3(3-X)^2) \cap (X^2(1+X)^2 + Y(3-X))) \\ &\overset{7)}{=} 4 + I(P, (X^4(1+X)^4 - X^3(3-X)^2) \cap (X^2(1+X)^2 + Y(3-X))) \end{aligned}$$

Now the first curve is of the form $-9X^3 + X^4(\ldots)$. Therefore, as in previous points and as we are in the case $\operatorname{char} k \neq 3$, we can factor out X^3 and discard the rest, so

$$\begin{aligned} I(P, C \cap E) &= 4 + I(P, X^3 \cap (X^2(1+X)^2 + Y(3-X))) \\ &\overset{6)+7)}{=} 4 + 3I(P, X \cap Y) = 7. \end{aligned}$$

$C \cap F$: We have

$$\begin{aligned} I(P, C \cap F) &\overset{7)}{=} I(P, (Y^2 - X^3) \cap ((X^2 + X^3)^3 - 4X^5) \\ &= I(P, (Y^2 - X^3) \cap ((\ldots)X^7 + X^6 - 4X^5). \end{aligned}$$

If $\operatorname{char} k \neq 2$ then by 5) we have $I(P, C \cap F) = 10$, and if $\operatorname{char} k = 2$ then by 5) we have $I(P, C \cap F) = 12$.

$D \cap E$: The tangents to D at P are $X \pm Y$, and the tangents to E at P are Y and $Y \pm \sqrt{3}X$. Hence if $\operatorname{char} k \neq 2$, D and E share no tangents at P, and thus by 5) we have

$$I(P, D \cap E) = m_P(D) m_P(E) = 2 \cdot 3 = 6.$$

Now if $\operatorname{char} k = 2$ we have $E = (X+Y)^2(X^2+Y^2+Y)$, and thus by 6) we obtain

$$\begin{aligned} I(P, D\cap E) &= I(P, D\cap (X+Y)^2) + I(P, D\cap (X^2+Y^2+Y)) \\ &\overset{6)+5)}{=} 2I(P, D\cap (X+Y)) + 2\cdot 1 \\ &\overset{7)}{=} 2I(P, X^3\cap (X+Y)) + 2 \\ &\overset{6)+7)}{=} 6+2 \\ &= 8. \end{aligned}$$

$D\cap F$: We have

$$\begin{aligned} I(P, D\cap F) &\overset{7)}{=} I(P, (Y^2-X^3-X^2)\cap ((2X^2+X^3)^3 - 4X^2(X^3+X^2))) \\ &= I(P, (Y^2-X^3-X^2)\cap (X^9+2(\ldots)-4X^4). \end{aligned}$$

If $\operatorname{char} k \neq 2$ then we may use 6) and 2) to obtain

$$\begin{aligned} I(P, D\cap F) &= I(P, (Y^2-X^3)\cap X^4) \\ &\overset{6)+7)}{=} 4I(P, Y^2\cap X) \\ &\overset{6)}{=} 8. \end{aligned}$$

If $\operatorname{char} k = 2$, a similar computation shows that $I(P, D\cap F) = 18$.

$E\cap F$: One way to make progress is always to make our curves reducible using 7) and then split up the computation using 6). Here, one way to do so here is to make a (X^2+Y^2) factor appear. This can be achieved by adding $Y\cdot E$ to F, giving

$$F + Y\cdot E = (X^2+Y^2)\underbrace{((X^2+Y^2)^2 + Y(X^2+Y^2) - Y^2)}_{F':=}.$$

Hence, using 7) and 6), we obtain

$$I(P, E\cap F) = I(P, E\cap (X^2+Y^2))) + I(P, E\cap F')$$

Now if $\operatorname{char} k \neq 2$, then the tangents of X^2+Y^2 are $X\pm iY$, and thus they are different from the tangents of E (Y and $X\pm Y$). Hence by 5) we obtain

$$I(P, E\cap (X^2+Y^2)) = 3\cdot 2 = 6.$$

On the other hand, if $\operatorname{char} k = 2$, then $E = (X^2+Y^2)(X^2+Y^2+Y)$, and so by 1) we obtain $I(P, E\cap (X^2+Y^2)) = \infty$, which then also gives $I(P, E\cap F) = \infty$, so we may assume $\operatorname{char} k \neq 2$ from now on.

To compute $I(P, E\cap F')$, we subtract E from F' to obtain

$$I(P, E\cap F') = I(E\cap (-Y(3X^2-Y^2) + Y(X^2+Y^2) - Y^2)).$$

As the second curve is divisible by Y, we use 6) and then 7) to obtain

$$I(P, E \cap F') = 4 + I(P, E \cap (-2X^2 + 2Y^2 - Y)).$$

By 7), we can replace every X^2 in E by $Y^2 - 2^{-1}Y$, giving

$$I(P, E \cap F') = 4 + I(P, ((2Y^2 - 2^{-1}Y)^2 + 3(Y^2 - 2^{-1}Y)Y - Y^3) \cap (-2X^2 + 2Y^2 - Y))$$

Now the first curve is $-(5/4)Y^2 + 4Y^4$. So if $\operatorname{char} k \neq 5$ we can factor out Y^2 and discard the rest by 6) and 2), giving

$$\begin{aligned} I(P, E \cap F') &= 4 + I(P, Y^2 \cap (-2X^2 + 2Y^2 - Y)) \\ &\overset{6)+7)}{=} 4 + 2I(P, Y \cap X^2) \\ &= 8. \end{aligned}$$

If $\operatorname{char} k = 5$, then a similar computation shows that $I(P, E \cap F') = 12$.

Piecing everything together, we obtain $I(P, E \cap F) = \infty$ if $\operatorname{char} k = 2$, $I(P, E \cap F) = 18$ if $\operatorname{char} k = 5$ and $I(P, E \cap F) = 14$ if $\operatorname{char} k \neq 2, 5$.

Exercise 4. Consider the affine curves $F = Y - X^2$ and $L = aY + bX + c$, where $a, b, c \in k$ and $(a, b) \neq (0, 0)$.

(1) Compute the intersection points $P \subseteq F \cap L$ and their intersection numbers $I(P, F \cap L)$. Consider $s = \sum_P I(P, F \cap L)$. Give a necessary and sufficient condition for $s = 1$.

Let us identify $\mathbb{A}^2_k$ with the affine open subset $U_1 = \{x_1 \neq 0\} \subseteq \mathbb{P}^2_k$, where we use projective coordinates x_1, x_2, x_3. Consider $\overline{V}$ (resp. $\overline{L}$) the closure of $V(F) \subseteq U_1$ (resp. $V(L)$) in $\mathbb{P}^2_k$.

(2) Assume that $s = 1$. Show that $\overline{V}$ and $\overline{L}$ admit another intersection point outside U_1 and that the intersection number (computed in the affine plane U_2 or U_3) is 1.

(3) Same questions with $F = XY - 1$.

Solution 4.

(1) Note that

$$k[X, Y]\big/(Y - X^2, aY + bX + c) \cong k[X]\big/(aX^2 + bX + c)$$

and the latter is spanned by $1, X$, so by 9) we have $s \leq 2$.

Note that $0 = Y - X^2 = aY + bX + c \Rightarrow aX^2 + bX + c = 0$. If $a \neq 0$, $b^2 - 4ac \neq 0$, there are two simple points

$$P_i = \left(x_i, x_i^2\right)$$

where x_1, x_2 are the roots of $aX^2 + bX + c$, so as $s \leq 2$ they must both be simple. If $b^2 - 4ac = 0$ there is a double point: indeed, in this case the intersection point is $P = (-b/2a, b^2/4a^2)$, and then

$$F^P = Y + \frac{b^2}{4a^2} - \left(X - \frac{b}{2a}\right)^2 = Y - X^2 + \frac{b}{a}X$$
$$L^P = a\left(Y + \frac{b^2}{4a^2}\right) + b\left(X - \frac{b}{2a}\right) + c = aY + bX$$

so

$$I(P, F \cap L) = I(0, F^P \cap L^P) \overset{7)}{=} I(0, X^2 \cap (aY + bX))$$
$$\overset{6)}{=} 2I(0, X \cap (aY + bX)) \overset{7)}{=} 2.$$

If $a = 0$ there is one simple point, given by $P = (-c/b, c^2/b^2)$. Indeed, we have

$$F^P = Y + \frac{b^2}{c^2} - \left(X - \frac{b}{c}\right)^2 = Y - X^2 + \frac{2b}{c}X$$
$$L^P = b\left(X - \frac{b}{c}\right) + c = bX,$$

so they don't share a tangent and hence $I(P, F \cap L) = 1$. Hence

$$s = 1 \iff a = 0$$

.

(2) By (1) we have $a = 0$. We denote the coordinates of $\mathbb{P}^2$ by X, Y, Z, so that $\mathbb{A}^2$ is identified with $\{Z \neq 0\}$. The intersection point in $\{Z \neq 0\}$ is then $[-c/b, c^2/b^2 : 1] = [-bc : c^2 : b^2]$.

To compute $\overline{V}$ and $\overline{L}$ we homogenize the equations to obtain

$$\overline{V} = V_p(YZ - X^2), \quad \overline{L} = V_p(bX + cZ).$$

As we already computed all the intersection points in $\{Z \neq 0\}$, the remaining intersection points must have $Z = 0$, and then it also follows that $X = 0$, and $[0 : 1 : 0]$ is indeed in the intersection. Hence this is the unique intersection point outside of $\{Z \neq 0\}$. To compute the intersection multiplicity, we dehomogenize by setting $Y = 1$, and then for $P = (0, 0)$ obtain

$$I(P, (Z - X^2) \cap (bX + cZ)) = 1$$

by 5).

(3) Note that

$$k[X, Y] \big/ (XY - 1, aY + bX + c) \cong k[X, X^{-1}] \big/ (aX^{-1} + bX + c)$$

and the latter is spanned by $1, X$, so we obtain $s \leq 2$ by 9).

If $XY = 1$ and $aY + bX + c = 0$, then multiplying with X we obtain $bX^2 + cX + a = 0$. If $b \neq 0$ and $a \neq 0$, you can check as in the previous parts that we obtain either two solutions which thus give simple points, or a double point. If $b = 0$ then we only obtain one solution, and by symmetry the same holds for $a = 0$. As in the previous points, you can check that F and L share no tangents in this case, and thus the intersection multiplicity is 1.

In summary, we obtain that $s = 1$ if and only if $a = 0$ or $b = 0$.

By symmetry, we assume $a = 0$; the case $b = 0$ follows by exchanging X and Y. Note that$\bar{V} = V_P(XY - Z^2)$ and $\bar{L} = V_P(bX + cZ)$. As we already computed the intersection points in $\{Z \neq 0\}$, the remaining ones must have $Z = 0$, which then gives also $X = 0$, so that we obtain the intersection point $[0 : 1 : 0]$. To compute the multiplicity, we dehomogenize by setting $Y = 1$, and then for $P = (0, 0)$, we obtain as in point (2) that $I(P, (Z - X^2) \cap (bX + cZ)) = 1$.

Remark. The point of the exercise is the following: you can convince yourself that now matter the value of $s = \sum_{P \in F \cap L} I(P, F \cap L)$, if we take the closure in $\mathbb{P}^2$ and add the intersection multiplicities for the 'new' intersection points to s as well, we always obtain 2 (e.g. if $s = 1$, then in $\mathbb{P}^2$ we obtained precisely one other simple point of intersection). This is an instance of Bezout's theorem, which we will see later in the course: if $F, G \subseteq \mathbb{P}^2$ are projective curves of degree d resp. e with no common components, then we have $\sum_{P \in F \cap G} I(P, F \cap G) = de$. That is, the number of intersection points, counted with the appropriate multiplicity, is always just de.

Exercise 5. Let F be an affine plane curve. Let L be a line that is not a component of F. Suppose that $L = \{(a + tb, c + td),\ t \in k\}$. Define $G(T) = F(a + Tb, c + Td)$ and consider its factorization $G(T) = \epsilon \prod_i (T - \lambda_i)^{e_i}$ where the λ_i are distinct.

(1) Show that there is a natural one-to-one correspondence between the λ_i and the points $P_i \in L \cap F$.
(2) Show that, under this correspondence, $I(P_i, L \cap F) = e_i$. In particular, $\sum_i I(P_i, L \cap F) \leq \deg(F)$ (see for instance exercise 4).

Solution 5.

(1) We have $t = \lambda_i \iff F(a + \lambda_i b, c + \lambda_i d) = 0$, so if we denote $\tau \colon \mathbb{A}^1 \to \mathbb{A}^2$ the map sending $t \mapsto (a + tb, c + td)$, then τ induces a bijection between $\{\lambda_i\}_i$ and $F \cap L$.
(2) Let $L = dX - bY - ad + bc$ be the function defining L. Then the map

$$k[X, Y] \mapsto k[T]$$
$$X \mapsto a + Tb,\ Y \mapsto c + Td$$

induces an isomorphism $k[X, Y]/(F, L) \cong k[T]/G$. If $P \in F \cap L$, then by point (1) we have $P = \tau(\lambda_j)$ for some j. Under the isomorphisms, the maximal ideal $\mathfrak{m}_P = (X - P_x, Y - P_y)$ is mapped to $(T - \lambda_j)$. If we localize both sides at these ideals, we obtain

$$\mathcal{O}_P(\mathbb{A}^2)\Big/(F,L) \cong k[T]_{(T-\lambda_i)}\Big/(\prod_i (T-\lambda_i)^{e_i}) = k[T]_{(T-\lambda_i)}\Big/((T-\lambda_j)^{e_j})$$

so that $I(P_i, F\cap L) = e_i$. As $\deg G \leq \deg F$, we conclude by 9) that $\sum_i I(P_i, L\cap F) \leq \deg F$.